FACULTY OF SCIENCE

MASTER PROGRAM OF MATHEMATICS

A STUDY ON SOME FORMS OF PROJECTIVITY AND INJECTIVITY

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Dedication

To everyone who have ever taught me, supported me and encouraged me, to my supportive family, my loving friends, and my teachers who guided and supervised me especially Prof. Mohammed Saleh, Dr. Hasan Yousef and Dr. Khaled Al-Takhman. I thankfully dedicate this thesis to you and give special thanks to my parents.

Abstract

In the category of R-modules over rings many authors studied the concepts of projective and injective modules and its generalization. Saleh, M. studied weak projectivity and weak injectivity over R- modules. In this thesis, we study some forms of projectivity and injectivity of semimodules over semirings especially we generalize the concept of weakly projective module for weakly projective semimodule and we introduce some of its basic characteristics which are analogous to ring theory also we study its dual concept which is weakly injective semimodule and we study some of its related properties in a similar manner.

Keywords: Semiring; Semimodule; Projective semimodule; Injective semimodule; Weakly projective semimodule; Weakly injective semimodule.

الملخص

في هذه الرسالة، ندرس بعض أشكال الإسقاط والحقن لشبه الوحدات على شبه الحلقات والتي تعتبر تعميماً للوحدات على الحلقات، وتحديداً تعميم مفهومي شبه الإسقاط الضعيف و شبه الحقن الضعيف ودراسة بعض الخصائص الأساسية المتعلقة بهما.

الكلمات المفتاحية: شبه الحلقات، شبه الوحدات، الوحدات شبه الإسقاطية، وحدات شبه الحقن، الوحدات شبه الإسقاطية الضعيفة، وحدات شبه الحقن الضعيفة.

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LIST OF SYMBOLS

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integer numbers
$\leq$	S-subsemimodule
$\ll$	Small subsemimodule
$\trianglelefteq$	Large subsemimodule
$\cong$	Isomorphic
$\simeq$	Semiisomorphic
$\oplus$	Direct sum
$\coprod$	Coproduct
$\twoheadrightarrow$	Surjection
U/V	Quotient semimodule over V
$\mathcal{M}_S$	Category of S-semimodules
$\mathcal{P}(U)$	Projective cover of semimodule U
$H(U)$	Injective hull (envelope) of semimodule U
$i_V : V \to U$	Natural inclusion, where $V \leq U$
$\pi_V : U \to U/V$	Natural projection, where $V \leq U$
U^V	the set of all maps from V to U

CHAPTER 1

PRELIMINARIES

1.1 Introduction

Semirings considered as a common generalization of rings which have importance in different fields in mathematics and computer science. The definition of semirings was first appear explicitly in 1934 by Vandiver. Semiring is defined simply as a ring without negative elements (without the requirement of additive inverse).

The definition of semimodules was introduced during the period of 1981 - 1990 in a series of several papers by M. Takahashi. The construction of semimodules over semirings is corresponding to the construction of module over ring which has a great role in characterizing properties of the semiring and it can be defined simply like a module over a ring except that it is only a commutative monoid rather than an abelian group.

The concepts of 'injectivity' and 'projectivity' of objects on R-modules were studied in [5]. Dually these two concepts introduced in [1] for semimodules over semirings.

In 1990, weak injective modules were studied in [1] by S.K. Jain and S.R. López-Permouth. After that many properties related to this concept were studied by them in [12] that we will generalize some of its results for semiring theory in this research.

In 1993, S.K. Jain, S.R. López-Permouth and Saleh, M. studied a dual concept of weakly injective modules which is weakly projective modules in [13] that we will also generalize some of its results for semiring theory in this research.

Generalizing the last two mentioned concepts for semiring theory based on two basic concepts needed for this work "projective cover" dually "injective envelope" which were studied recently in 2014 - 2016 by S.N. Il'in in [10], [11] respectively.

Our basic reference in semiring theory is Golan's book [9] - "Semirings and their Applications". We assume in this thesis that all semimodules are right and unitial.

Our thesis consists of three main chapters. In chapter 1, we review the most important definitions, results and theorems in semiring theory that we need later in this thesis. In chapter 2, we present the concept of weak relative projectivity of right S-semimodule, also we study some properties related to the concept. In chapter 3, we introduce the dual concept of weak relative projectivity which is weak relative injectivity and dualize most of the related properties.

In this chapter, we review basic results and definitions which are helpful later in next chapters. Mainly depending on Golan's book [9] as a needful reference.

1.2 Basics in Semimodules

Basic definitions and results in semimodule theory are provided in this section.

Definition 1.1. (Semigroup). [9] An algebraic structure $(S, *)$ with a non-empty set S and an operation $*$ is called semigroup if it satisfies the following properties:

1. The operation $*$ is binary that is $u * v \in S$ for all $u, v \in S$.

2. The operation $*$ associative that is $(u * v) * w = u * (v * w)$ for all u, v and $w \in S$.

Example 1.2. The sets of real, integer and complex numbers under multiplication are semigroups.

Definition 1.3. (Monoid). [9] A monoid $(M, *)$ is a semigroup with an identity element that is there exists e in M such that for every element $a \in M$ the equation $a * e = e * a = a$ holds. Therefore, the monoid is characterized by the triple $(M, *, e)$.

Definition 1.4. A commutative monoid (an abelian monoid) is a monoid whose operation is commutative.

Example 1.5. • The set of natural numbers under addition $(\mathbb{N}, +)$, is a commutative monoid.

• Integer numbers under multiplication $(\mathbb{Z}, \times)$ is a commutative monoid with identity element one.

Definition 1.6. A group $(G, *)$ is a monoid with an inverse element that is there exists a in G such that for every element $b \in G$ the equation $a * b = b * a = e$.

Definition 1.7. (Ring). [5] A ring $(R, +, .)$ is a nonempty set R and two binary operation addition + and multiplication . satisfying the following conditions:

1. $(R, +, 0)$ is an abelian group;
2. $(R, \cdot)$ is a semigroup;
3. Multiplication distributes over addition from both sides;

If there exists a multiplicative identity then we say that R is a ring with unity.

Definition 1.8. (Module). [9] A right Module over a ring R is an abelian group $(M, +, 0_M)$ with a map $M \times R \mapsto M$ denoted by $(m, r) \mapsto mr$ which called scalar multiplication in which the following conditions satisfied for all $m_1, m_2 \in M, r_1, r_2 \in R$:

1. $(m_1r_1)r_2 = m_1(r_1r_2)$;
2. $(m_1 + m_2)r_1 = m_1r_1 + m_2r_1$;
3. $m_1(r_1 + r_2) = m_1r_1 + m_1r_2$;
4. $m_11 = m_1$;

Definition 1.9. (Semiring). [9] A semiring (S, +, .) is a nonempty set S and two binary operation addition + and multiplication . satisfying the following conditions:

1. $(S, +, 0)$ is a commutative monoid;
2. $(S, \cdot, 1)$ is a monoid;
3. Multiplication distributes over addition from both sides;

4. $1 \neq 0$. (That is the case when $0 = 1$, and so $S = \{0\}$, is excluded).

5. $0s' = s'0 = 0$ for all $s' \in S$.

Example 1.10. [9] A bounded distributive lattice $D = \langle D, \vee, 0, \wedge, 1\rangle$ such that $\wedge$ distributes over $\vee$ that is $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ for all $\alpha, \beta \in D$ where 0 is the unique minimal element and 1 is the unique maximal element, D is a semiring.

Definition 1.11. (Semimodule). [9] A right semimodule over a semiring S is a commutative monoid $(M, +, 0_M)$ with a map $M \times S \mapsto M$ denoted by $(m, s) \mapsto ms$ which called scalar multiplication in which the following conditions satisfied for all $m_1, m_2 \in M, s_1, s_2 \in S$:

1. $(m_1 s_1)s_2 = m_1(s_1 s_2)$;

2. $(m_1 + m_2)s_1 = m_1 s_1 + m_2 s_1$;

3. $m_1(s_1 + s_2) = m_1 s_1 + m_1 s_2$;

4. $m_1 1 = m_1$;

5. $m_1 0_s = 0_M s_1 = 0_M$.

In the same way left S-semimodules are defined. Any semiring is a right and left semimodule over itself.

We denote the category of right S-semimodules by $\mathcal{M}_S$.

Definition 1.12. (Subsemimodule). [9] A nonempty subset A of a right S-semimodule B is a subsemimodule of B if and only if it is closed under addition and scalar multiplication which denoted by $A \leq B$.

Example 1.13. If X is a subset of a right S-semimodule Y then the intersection of all subsemimodules of Y which contains X is a subsemimodule of Y which is the subsemimodule **generated by** X. This semimodule is just $XS = \{x_1s_1 + \cdots + x_ns_n | x_i \in X, s_i \in S\}$. If Y is the subsemimodule generated by X then X is said to be **generating set** of Y, that is X is a **set of generators** for Y If X generates all of the semimodule Y. A right S-semimodule is **finitely generated** if it has a finite set of generators .

Definition 1.14. (Subtractive). [9] A nonempty subset X of a right S-semimodule Y is subtractive if and only if $x_1 + x_2 \in X$ and $x_1 \in X$ imply that $x_2 \in X$ for all $x_1, x_2 \in Y$.

Definition 1.15. (Homomrphism). [9] Let U, V be a right S-semimodules of a semiring S, then a function $\alpha : U \to V$ is an S-homomrphism if and only if the following holds:

1. $\alpha(u_1 + u_2) = \alpha(u_1) + \alpha(u_2)$ for all $u_1, u_2 \in U$.

2. $\alpha(us) = (\alpha(u))s$ for all $u \in U$ and $s \in S$.

- **Image** of α is $Im(\alpha) = \{\alpha(u) \mid u \in U\}$ which is a subsemimodule of V. Since if $\alpha(u_1), \alpha(u_2) \in Im(\alpha)$ then $\alpha(u_1) + \alpha(u_2) = \alpha(u_1 + u_2)$ and so $\alpha(u_1) + \alpha(u_2) \in Im(\alpha)$ since $u_1 + u_2 \in U$, that is $Im(\alpha)$ is closed under addition. Now, $\alpha(u_1)s = \alpha(u_1s)$ where $u_1s \in U$ and so $\alpha(u_1)s \in Im(\alpha)$ that is $Im(\alpha)$ is closed under scalar multiplication.

- **kernel** of α is $ker(\alpha) = \alpha^{-1}\{0_V\}$ which is subtractive subsemimodule of V. Since if $u_1 + u_2 \in Ker(\alpha)$ and $u_1 \in Ker(\alpha)$ then $\alpha(u_1) + \alpha(u_2) = \alpha(u_1 + u_2) = 0$ but $\alpha(u_1) = 0$ and so $\alpha(u_2) = 0$ that is $u_2 \in Ker(\alpha)$

Definition 1.16. [9] Let S be a semiring and let $\{U_i \mid i \in D\}$ be a family of right S-semimodules. The **external direct sum** of the S-semimodules

U_i is $\otimes_{i\in D}U_i = \{\langle u_i\rangle \mid u_i \neq 0$ for only finitely many indices $i\}$ under componentwise addition and scalar multiplication. As a special case for two semimodules U, V we define their **external direct sum** as follows:

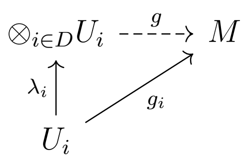
$$U \oplus V = \{(\alpha, \beta) \mid \alpha \in U \textit{ and } \beta \in V\}$$

with componentwise addition and scalar multiplication.

Remark 1.17. Let $\{U_i \mid i \in D\}$ be a family of right S-semimodules and U be a right S semimodule then, for each t in D we have canonical S-homomorphisms $\lambda_t : U_t \to \otimes U_i$, defined by $\lambda_t(u_i) = \langle v_i\rangle$, where

$$v_i = \begin{cases} 0 & if \quad i \neq t \\ u_t & if \quad i = t \end{cases}. \tag{1.1}$$

• If we have an S-homomorphism $g_i : U_i \to U$ for each $i \in D$ then there exists a unique S-homomorphism $g : \otimes_{i\in D}U_i \to U$ such that $g_i = g\lambda_i$ for each $i \in D$.



Definition 1.18. [9] If $\{U_i, i \in I\}$ is a family of S-subsemimodules of an S-semimodule U, then U is the **internal direct sum** of the subsemimodules U_i written $U = \oplus_{i\in I}U_i$ if and only if each element $u \in U$ can be written uniquely as $\sum u_i$, where $u_i \in U_i$ for each $i \in I$, that is $U = \sum u_i$ and $U_j \cap (\sum_{i\neq j} U_i) = 0$.

Definition 1.19. (Direct summand). [9] Let S be a semiring, then an S-semimodule U is called a direct summand of an S-subsemimodule V if and only if there is an S-subsemimodule V' of U such that $U = V \oplus V'$.

• **Notation**: For an arbitrary index set I the external direct sum of I copies of a semimodule U is denoted by $U^{(I)}$. Also, U^n represent the external direct sum of n factors of U.

Example 1.20. If U is a right S-semimodule generated by a subset V then we have a surjective S-homomorphism $S^{(V)} \to U$. We always have a surjective S-homomorphism from $S^{(U)}$ to U.

Definition 1.21. (Congruence relation). [9] Let U be a right S-semimodule. An equivalence relation γ on U is an S-congruence relation if and only if $u_1 \gamma u_2$ and $v_1 \gamma v_2$ in U then $(u_1 + v_1) \gamma (u_2 + v_2)$ and $u_1 s \gamma u_2 s$ for all $s \in S$.

Definition 1.22. (Factor semimodule). [9] Let γ be an S-congruence relation on U and, for each $u \in U$, let u/γ be the equivalence class of m with respect to this relation. Set $U/\gamma = \{u/\gamma \mid u \in U\}$ and define operations of addition and scalar multiplication on U/γ by setting $(u/\gamma)+(v/\rho) = (u+v)/\gamma$ and $(u/\gamma)s = (us)/\gamma$ for all $u, v \in U$ and $s \in S$ Then U/γ is an S-semimodule, called the factor semimodule of M by γ. Moreover, we have a surjective S-homomorphism $U \to U/\gamma$ defined by $u \mapsto u/\gamma$.

Example 1.23. If V is a subsemimodule of a semimodule U, then V induces an S-congruence relation $\equiv_V$ on U, called the **Bourne relation**, defined as $u \equiv_V u'$ if and only if there exists elements $v, v' \in V$ such that $u+v = u'+v'$ and we denote the equivalence class of u by $u + V$ and the collection of all equivalence classes by U/V instead of $U/\equiv_V$. Also, if $u_1 + V = u_2 + V$ then there exists elements $a, b \in V$ such that $u_1 + a = u_2 + b$.
Moreover, we have a surjective S-homomorphism $\pi_V : U \to U/V$ defined by $u \mapsto u/V$.

Definition 1.24. [9] If X, Y, X' and Y' are right S-semimodules then an S-homomorphism $f : X \to Y$ is:

1. **Monomorphism** if and only if for any S-homomorphisms $g, g' : X' \to X$ then $fg \neq fg'$.

2. **Epimorphism** if and only if for any S-homomorphisms $g, g' : Y \to Y'$ then $gf \neq g'f$.

3. **Surjective (onto, or epic)** if for each element $y \in Y$ there is element $x \in X$ such that $f(x) = y$, that is $Im(f) = Y$.

4. **Injective (one to one, or monic)** if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

5. **Isomorphism** if it is both surjective and injective.

6. A surjective S-homomorphism having kernel $\{0\}$ is an **semi-isomorphism**.

Prposition 1.25. If $f : X \to Y$ is an S-homomorphism between right S-semimodules then:

1. f is monic if and only if it is an S-monomorphism.

2. If f is monic then $ker(f) = 0$, but the converse is not necessary true.

3. If f is surjective then it is an epimorphism and $f(X)$ is subtractive, but the converse is not true.

4. if f is an epimorphism and $f(X)$ is subtractive then f is surjective.

Now, we prove (1) - (4) then we give counter examples for (2), (3).

Proof. 1. Let $f : X \to Y$ be an S-homomorphism of right S-semimodules. If f is monic, it is clearly a monomorphism. Now, by contrapositive If f is not monic then there exist elements $x \neq x'$ of X satisfying $f(x) = f(x')$. Define S-homomorphisms $g, g' : S \to X$ where S considered as a right semimodule over itself, by setting $g(s) = xs$ and $g'(s) = x's$. Then $g \neq g'$ but $fg(s) = f(xs) = f(x)s = f(x')s = f(x's) = fg'(s)$ for all $s \in S$, showing that f is not a monomorphism.

2. Let $x \in ker(f)$ then $f(x) = 0 = f(0)$ but f is monic then $x = 0$ so $ker(f) = 0$.

3. Let $g, g' : Y \to Y'$ such that $gf(X) = g'f(X)$ but f is surjective and so $f(X) = Y$ then $g(Y) = g'(Y)$ so f is an epimorphism. Moreover, let $a, b \in Y$ where a and $a + b \in f(X)$ since f is surjective $f(X) = Y$ and so $b \in f(X)$ that is $f(X)$ subtractive subsemimodule of Y.

4. Assume that f is an epimorphism in which $f(X)$ is subtractive, but that f is not surjctive. Set $Y' = f(X)$. Then we have S-homomorphisms $g, g' : Y \to Y/Y'$ given by $g(y) = 0/Y'$ and $g'(y) = y/Y'$, Moreover, $gf = g'f$. Since f is not surjective, there exists an element $y \in Y \setminus Y'$. Also, $g(y) = g'(y)$ and so $y/Y' = 0/Y'$ implies that there exists elements $f(x), f(x') \in Y'$ such that $y + f(x) = y' + f(x') \in Y'$ but $f(X)$ subtractive so $y \in Y'$ this is a contradiction and so $g'(y) = y/Y' \neq 0/Y' = g(y)$ so $g \neq g'$ this contradicts that f is an epimorphism.

□

As we stated in the previous proposition kernel of a function may equal zero but the function is not monic, see the following example:

Example 1.26. [9] Consider the totally ordered set $R = \{0, a, 1\}$ where addition is defined to be max and multiplication to be min and let $\mathbb{B} = \{0, 1\}$ where $1 + 1 = 1$. Where R, $\mathbb{B}$ are semirings by (Example 1.10) and considered as semimodules over themselves. Let $f : S \to \mathbb{B}$ be the character of f defined by $f(0) = 0$ and $f(a) = f(1) = 1$, so $ker(f) = \{0\}$ but f is not monic.

Also, we stated that not every epimorphism is surjective, see the following example:

Example 1.27. Consider the inclusion map $i : Z^+ \to Z$ is an epimorphism of Z^+ semimodules since if $f, g : Z \to M$ then $f \circ i(Z^+) = g \circ i(Z^+)$ then $f(Z^+) = g(Z^+)$ and so $f = g$, but $i(Z^+) = Z^+$ not Z and so i is not surjective.

Remark 1.28. Let $\alpha : U \to V$ be a homomorphism of S-semimodules. α is an isomorphism if and only if there exists a homomorphism of S-semimodules $\beta : V \to U$ such that $\alpha\beta = id_V$ and $\beta\alpha = id_U$.

Corollary 1.29. [9] Let $f : X \to Y$ be a surjective S-homomorphism of right S-semimodules. Then there exists an S-semiisomorphism $X/ker(f) \to Y$, denoted by $X/ker(f) \simeq Y$.

Definition 1.30. [7] Let Y be an S-semimodule a subsemimodule X of Y is called:

- Essential (large) in Y, denoted by $X \trianglelefteq Y$, if $X \cap Z = 0$ implies $Z = 0$, for any subsemimodule $Z \leq Y$.
- Superfluous (small) in Y, denoted by $X \ll Y$, if $X + Z = Y$ implies $Z = Y$, for any subsemimodule $Z \leq Y$.

Definition 1.31. [7] Let X, X' be subsemimodules of Y, X' is called:

- Y-complement of X, if X' is maximal with respect to $X \cap X' = 0$. Moreover, every subsemimodule of Y has an Y-complement using the maximal principle if $X \leq Y$, then the set of subsemimodules of Y whose intersection with X is zero contains a maximal element X'.

- Y-supplement of X if X' is minimal proper subtractive subsemimodule with respect to $X' + X = Y$. In general, not every subsemimodule has a supplement and a semimodule in which all its subsemimodules have supplements is called supplemented semimodule.

Lemma 1.32. [6] Let M be an S-semimodule. Then if L and K are subtractive subsemimodules of M, then $L + K = \{u + v : u \in L, v \in K\}$ is a subtractive subsemimodule of M.

Prposition 1.33. [7] Let B be a semimodule over a semiring S and A, A' are subtractive subsemimodules of B with A' an B-complement of A. Then, $A \oplus A' \trianglelefteq B$.

Proof. Want to show that $A \oplus A' \trianglelefteq B$, that is if $(A \oplus A') \cap Z = 0$ then $Z = 0$ (where $Z \leq B$) assume by contradiction that $Z \neq 0$ and $(A \oplus A') \cap Z = 0$. Now, we show that $A \cap (A' \oplus Z) = 0$. Let $a \in A \cap (A' \oplus Z)$ want to show that $a = 0$ let $a = a' + z$ for some $a \in A, a' \in A'$ and $z \in Z$. Now, $a' + z \in A \oplus A'$ since $a \in A \oplus A'$ and so $z \in A \oplus A'$ since $a' \in A \oplus A'$ and $A \oplus A'$ is a subtractive subsemimodule of B by (Lemma 1.32). Therefore $z \in (A \oplus A') \cap Z = 0$ and $a = a' \in A \cap A' = 0$ (since A' is B-complement of A) so $a = 0 \implies A \cap (A' \oplus Z) = 0$ but $A' \oplus Z \supseteq A'$ and this contradicts the maximality of A'. Thus, $Z = 0$ and $A \oplus A' \trianglelefteq B$. □

1.3 Steadiness

In this section we present a basic concept and some results related to it which play an important role in our thesis.

Remark 1.34. [9] Each S-homomorphism of semimodules $\sigma : U \to V$ defines a congruence relation $\equiv_\sigma$, on U by setting $u \equiv_\sigma u'$ where u, u' are elements of U if and only if, $\sigma(u) = \sigma(u')$. Another congruence relation defined on U by σ is the relation $\equiv_{ker(\sigma)}$, by setting $u \equiv_{ker(\sigma)} u'$ if and only if, there exists elements $e, e' \in ker(\sigma)$ satisfying $u + e = u' + e'$. If u, u' satisfying $u \equiv_{ker(\sigma)} u'$ then surely $u \equiv_{\sigma} u'$, but the converse does not necessarily true.

Definition 1.35. [9] If $\sigma : X \to Y$ is an S-homomorphism and the relations $x \equiv_{\sigma} x'$ and $x \equiv_{ker(\sigma)} x'$ coincide where $x, x' \in X$, then the S-homomorphism σ is **steady**.

Remark 1.36. A steady S-homomorphism $\sigma : U \to V$ is monic if and only if $ker(\sigma) = 0$. Also, by (Corollary 1.29) if σ is a steady surjective morphism of semimodules then V is S-isomorphic to $U/ker\sigma$, denoted by $V \cong U/ker\sigma$.

Lemma 1.37. Let $\{f_{\alpha} : K_{\alpha} \to M_{\alpha}\}_A$ be a family of right S-semimodule morphisms and consider the S-homomorphism

$$f : \bigoplus_{\alpha\in A} K_{\alpha} \Longrightarrow \bigoplus_{\alpha\in A} M_{\alpha}$$

Then f is steady if and only if f_{α} is steady for every $\alpha \in A$.

Lemma 1.38. Suppose that N, M, and Q are S-semimodules, where $h : Q \to N$ surjective homomorphisms and $f : N \to M$ is a homomorphism. Then:

1. If $f \circ h$ is steady homomorphism, then f is steady homomorphism.
2. Assume that h is steady. Then f is steady if and only if $f \circ h$ is steady.

Proof. (1) Suppose $f \circ h$ is steady. Assume $f(n_1) = f(n_2)$ for some n_1, n_2 belongs to N. Since h is surjective, then $(f \circ h)(q_1) = (f \circ h)(q_2)$ for some q_1, q_2 belongs to Q. By assumption, $f \circ g$ is steady and so there exist k_1, k_2 belongs to $ker(f \circ h)$ such that $q_1 + k_1 = q_2 + k_2$ whence $n_1 + h(k_1) = n_2 + h(k_2)$, but $h(k_1), h(k_2)$ belongs to $ker(f)$. Thus, f is steady.

(2) Suppose h and f are steady. Assume $(f \circ h)(q_1) = (f \circ h)(q_2)$ for some q_1, q_2 belongs to Q. Since f is steady, then $h(q_1)+k_1 = h(q_2)+k_2$ where k_1, k_2 belongs to $ker(f)$. But, h is surjective, whence $k_1 = h(q'_1)$ and $k_2 = h(q'_2)$ where q'_1, q'_2 belongs to Q, i.e. $h(q_1 + q'_1) = h(q_2 + q'_2)$. Since h is steady, $q_1 + q'_1 + k'_1 = q_2 + q'_2 + k'_2$ where k'_1, k'_2 belongs to $ker(h)$ but, $q'_1 + k'_1, q'_2 + k'_2$ belongs to $ker(f \circ h)$. We conclude $f \circ h$ is steady. □

Prposition 1.39. [9] Let $f : U \to V$ be an S-homomorphism between right S-semimodules and $g : U \to W$ be a surjective steady S-homomorphism between right S-semimodules where $ker(g) \subseteq ker(f)$. Then:

1. A unique S-homomorphism $h : W \to V$ exists and satisfy $f = hg$;
2. If f is monic then h is monic;
3. $ker(h) = g(ker(f))$; and
4. $h(W) = f(U)$.

Proof. (1) Since g is surjective if $w \in W$ then $g^{-1}(w) \neq \varnothing$. If $u, u' \in g^{-1}(w)$ then $u \equiv_g u'$ and so, by steadiness, $u \equiv_{ker(g)} u'$. Therefore, there exist elements $e, e' \in ker(g) \subseteq ker(f)$ saisfying $u + e = u' + e'$ and so

$$f(u) = f(u) + f(e) = f(u + e) = f(u' + e') = f(u') + f(e') = f(u').$$

Now, define the function $h : W \to V$ by $h : w \to f(u)$, where u is any element of $g^{-1}(w)$. Then h is well-defined, and it is an S-homomorphism of semimodules satisfying $f = hg$. Moreover, if $h' : W \to V$ is an S-homomorphism satisfying $f = h'g$ and if $w \in W$ then for any $u \in g^{-1}(w)$ we have $h'(w) = h'g(u) = hg(u) = h(w)$, proving that $h = h'$.

(2) Assume f is monic. If $h(w_1) = h(w_2)$ and if $u_i \in g^{-1}w_i$ for $i = 1, 2$, then $f(u_1) = hg(u_1) = hg(u_2) = f(u_2)$ and so $u_1 = u_2$. Therefore $w_1 = g(u_1) = g(u_2) = w_2$, proving that h is monic.

(3) Clearly $g(ker(f)) \subseteq ker(h)$. Conversely, if $w \in ker(h)$ and if $u \in g^{-1}(w)$ then $f(u) = h(w) = 0_V$ so $w = g(u) \in g(ker(f))$.

(4) Straight from the definition. □

Prposition 1.40. [9] Let $f : U \to V$ be an S-homomorphism of right S-semimodules. Let $g : W \to V$ be a monic S-homomorphism between right S-semimodules such that $g(W)$ is a subtractive subsemimodule of W containing $f(U)$. Then:

1. A unique S-homomorphism $h : U \to W$ exists and satisfy $f = gh$;
2. $ker(h) = ker(f)$; and
3. h is monic if and only if f is monic.

Proof. (1) If $u \in U$ then $f(u) \in f(U) \subseteq g(W)$. Since g is monic, there exists a unique element w of W satisfying $h(w) = f(u)$. Set $h(u) = w$. By uniqueness, the function $h : U \to W$ is defined and satisfy $f = gh$, which is unique.

(2) If $u \in ker(f)$ then $g(0_W) = 0_V = f(u)$ so $h(u) = 0_W$, proving that $u \in ker(h)$. Conversely, if $u \in ker(h)$ then $f(u) = gh(u) = 0_V$ so $u \in ker(f)$.

(3) Immediately from the definition. □

1.4 Projective and Injective Semimodules

We review in this section basic results and definitions related to projective and injective semimodules.

Definition 1.41. [9] Let U be an S-semimodule. A subset $\{u_i\}_{i\in I}$ of U is called linearly independent if and only if, whenever $\sum_{j\in J} u_j s_j = 0$ we have $s_j = 0$ for all $j \in J$.

Definition 1.42. (Free). [9] A right S-semimodule X is free if X has a basis, that is a subset $\{e_j, j \in J\} \subseteq X$ which is a linearly independent generating set of X.

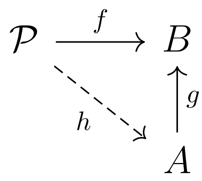
Prposition 1.43. [9] Suppose that S is a semiring and B is a right S-semimodule then there exists a free S-semimodule A and a surjective S-homomorphism from A to B.

Proof. Let B be a right S-semimodule. The result is trivial for the case of $B = \{0\}$. Let $B' = B\backslash\{0\}$ and let $A = S^{(B')}$. Let $\sigma : A \to B$ is defined by $\sigma : h \mapsto \sum_{b\in supp(h)} h(b)$. This is obvious a surjective S-homomorphism. □

Prposition 1.44. [9] Let A, B be right S-semimodules where B is free with basis N. For any $f \in A^N$, then there is a unique S-homomorphism $\gamma : B \to A$ satisfying that $\gamma(n) = f(n)$ for any $n \in N$.

Proof. Since B is free, any $b \in B$ can be written uniquely as $\sum_{n\in N} s_n n$ where $s_n \in S$ only finitely many of s_n's are nonzero. Let the function $\gamma : A \to B$ defined by $\sum s_n n \mapsto \sum s_n f(n)$. It is easy to show that γ is an S-homomorphism satisfying the desired property. Now, to show the uniqueness assume that $\rho : A \to B$ is an S-homomorphism satisfying that $\rho(n) = f(n)$ for all $n \in N$, then $\rho(\sum s_n n) = \sum s_n(\rho(n)) = \sum s_n f(n) = \sum s_n \gamma(n) = \gamma(\sum s_n n)$ and so $\gamma = \rho$. Thus γ is unique. □

Definition 1.45. [9] A right S-semimodule $\mathcal{P}$ is projective if and only if the following condition holds: if $g : A \to B$ is surjective S-homomorphism of right S-semimodules and $f : \mathcal{P} \to B$ is an S-homomorphism then there exists an S-homomorphism $h : \mathcal{P} \to A$ such that the following diagram commutes:



that is, $f = g \circ h$.

Prposition 1.46. [9] Every free right S-semimodule is projective.

Proof. Let X be a free right S-semimodule with basis E. Let $g : A \to B$ be a surjective S-homomorphism of right S-semimodules and let $f : X \to B$ be an S-homomorphism. Since g is surjective, for every element e of E there is a_e of A such that $g(a_e) = f(e)$. By (Proposition 1.44), there is a unique S-homomorphism $h : X \to A$ satisfying $h(e) = a_e$. Then $gh(e) = g(a_e) = f(e)$ for all $e \in E$ Thus, by Proposition 2.5, we have $f = gh$. □

Definition 1.47. [9] A right S-semimodule V is a retract of a right S-semimodule U if and only if there is S-homomorphisms $f : U \to V$ and $g : V \to U$ where f is surjective such that $f \circ g = i_V$.

Remark 1.48. If U and V are right S-semimodules and V is a direct summand of U then surely V is a retract of U. However, the converse is not true.

Prposition 1.49. [9] A right S-semimodule is a retract of a free right S-semimodule if and only if it is projective.

Proof. ⇒ If J is a projective right S-semimodule then, by (Proposition 1.43), there is a surjective S-homomorphism $\alpha : K \to J$ where K is free S-semimodule. Now, by projectivity there exists an S-homomorphism β :

$J \to K$ in which $\alpha\beta$ is the identity map on J.

$\Leftarrow$ Conversely, Let J be a retract of a free right S-semimodule K and let $\alpha : K \to J$ be a surjective S-homomorphism and $\beta : J \to K$ be an S-homomorphism in which $\alpha\beta$ is the identity map on J. Let $g : A \to B$ and $f : J \to B$ be an S-homomorphisms where g is surjective. Now, by projectivity of K by (Proposition 1.46), there exists an S-homomorphism $\sigma : K \to A$ in which $g\sigma = f\alpha$. Therefore $g\sigma\beta = f\alpha\beta$, and so $\sigma\beta : J \to A$ is a map having the property that is needed to prove projectivity of J. □

Corollary 1.50. A retract of a projective right S-semimodule is projective.

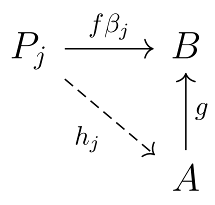
Proof. Straightforward from the previous proposition. □

Prposition 1.51. [9] If $\{P_j \mid j \in D\}$ is a family of right S-semimodules then $\coprod_{j \in D} P_j$ is projective if and only if each P_j is projective.

Proof. $\Rightarrow$ Assume $\coprod_{j \in D} P_j$ is projective. Now each P_j is a retract of $\coprod_{j \in D} P_j$ since each one of the $P_j{}'s$ is a direct summand of $\coprod_{j \in D} P_j$, and hence is projective. by the previous corollary.

$\Leftarrow$ Assume that P_j is projective for each $j \in D$. Now, for every $j \in D$, let $\alpha_j : \coprod_{j \in D} P_j \to P_j$ be the surjective S-homomorphism $\langle P_t \rangle \mapsto p_j$ and let $\beta_j : P_j \to \coprod_{j \in D} P_j$ be the inclusion map.

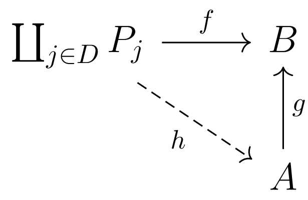
Let $g : A \to B$ be a surjective S-homomorphism of right S-modules and let $f : \coprod_{j \in D} P_j \to B$ be S-homomorphism. Then, since P_j's are projective for each $j \in D$ there exists an S-homomorphism $h_j : P_j \to A$ such that the following diagram commutes;



that is, $f\beta_j = gh_j$. Now define the S-homomorphism $h : \coprod_{j\in D} P_j \to A$ by $h : p \mapsto \sum_{j\in D} h_j\alpha_j(p)$. Then for $p \in \coprod_{j\in D} P_j$ we have

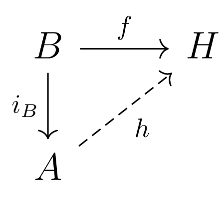
$$gh(p) = \sum_{j\in D} gh_j\alpha_j(p) = \sum_{j\in D} f\beta_j\alpha_j(p) = f(p)$$

and so, the diagram



commutes that is, $f = gh$. □

Definition 1.52. [9] Let H, A and B be a right S-semimodules H is injective if and only if, for a subsemimodule $B \leq A$, any S-homomorphism from B to H can be extended to an S-homomorphism from A to H.



that is, $f = h \circ i_B$.

Prposition 1.53. [9] Let J be an injective right S-semimodule. Then any direct summand of J is injective.

Proof. Let J' be a direct summand of J and let J'' be a subsemimodule of J satisfying $J = J' \oplus J''$. So, there is a surjective S-homomorphism $\phi : J \to J'$, the kernel of which is J''. Let $\psi : J' \to J$ be the inclusion map. If Y is a subsemimodule of a right S-semimodule X and if $f : Y \to J'$ is an S-homomorphism then, by injectivity there is an S-homomorphism $g : X \to J$ extending ψf in which the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi f} & J \\ {\scriptstyle i_Y}\downarrow & \nearrow_{g} & \\ X & & \end{array}$$

commutes that is, $\psi f = g i_Y$. In particular , if $y \in Y$ then $g(y) \in J'$ and so $\phi g(y) = \phi\psi f(y) = f(y)$, therefore $\phi g : X \to B'$ extends f in which the diagram,

$$\begin{array}{ccc} Y & \xrightarrow{f} & J' \\ {\scriptstyle i_Y}\downarrow & \nearrow_{\phi g} & \\ X & & \end{array}$$

commutes that is, $f = (\phi g) i_Y$ proving that J' is injective. $\square$

CHAPTER 2

WEAKLY PROJECTIVE SEMIMODULES

In this chapter, we present the concept of weakly projective S-semimodule that we will dualize in the next chapter. In section 1, we will state basic properties of projective covers. In section 2, we introduce the concept of weak relative projectivity of right S-semimodule. In section 3 we study some properties related to the concept of weak projectivity in semiring theory corresponding to ring theory.

2.1 Projective Covers of Semimodules

We present in this section the concept of projective covers and state some of its basic properties and study the relation between projective covers of semimodules.

Definition 2.1. [10] A surjective S-homomorphism $f : A \to B$ of S-semimodules is coessential if and only if for any S-homomorphism $g : C \to A$, if the map $f \circ g$ is surjective then g is surjective.

Definition 2.2. [10] Let X, $\mathcal{P}$ be a right S-semimodules, consider $\mathcal{P}$ together with a homomorphism $g : \mathcal{P} \to X$ then $\mathcal{P}$ is said to be projective cover of X if:

1. $\mathcal{P}$ is projective,

2. g is coessential.

We can see that, if S is a ring the above definition is equivalent to the usual one for modules over rings (since in this case if f is coessential then f is superfluous).

Remark 2.3. Each projective semimodule is projective cover of itself.

Proof. Assume X is projective semimodule $f : X \to X$ is coessential. Since for any S-homomorphism $g : Y \to X$, the map $f \circ g : Y \to X$ is surjective only when g is surjective. □

However, not all semimodules have projective covers.

Example 2.4. $\mathbb{Q}$ (set of rational numbers) has no projective cover as a Z-module since $\mathbb{Q}$ is not free and the only Z-modules that have projective cover are free modules.

Definition 2.5. [16] A surjective S-homomorphism of right S-semimodules $\gamma : U \to V$ is superfluous if $Ker\gamma \ll U$.

Definition 2.6. [16] Let $g : X \to Y$ be a homomorphism of right S-semimodules, g is called k-quasiregular if whenever $K \leq X$, $x \in X \backslash K$, $x' \in K$, and $g(x) = g(x')$ there exists $e \in \text{Ker } g$ such that $x = x' + e$.

Corollary 2.7. [16] Let $\beta : Y \to Z$ be a superfluous k-quasiregular homomorphism of right S-semimodules and let $\alpha \in Hom_S(X, Y)$. If $\beta\alpha$ is surjective, then α is surjective.

Proof. $\beta(Y) = Z = \beta(\alpha(X))$ since $\beta\alpha : X \to Z$ and $\beta : Y \to Z$ are surjective homomorphisms and β is k-quasiregular so the equality $\alpha(X) + Ker\beta = Y$ holds. Therefore, $\alpha(X) = Y$ since β is a superfluous S-homomorphism. □

Prposition 2.8. [16] Let X be a subtractive S-subsemimodule of a right S-semimodule Y and $X \ll Y$, then the canonical projection $P : Y \to Y/X$ is a superfluous surjective S-homomorphism.

Proof. We want to show that Ker p $\ll Y$, claim Ker p $= X$. Now, want to proof our claim:

$\Rightarrow X \subset$ Ker p, obviously.

$\Leftarrow$ Let $b \in$ Ker p there are $x_1, x_2 \in X$ satisfying $b + x_1 = 0 + x_2 = x_2$. Hence, $b \in X$. So, Ker p $\subset X$ and so our claim is proved. Thus, Ker p $= X \ll$ p and so, p is superfluous. □

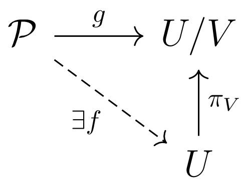
Remark 2.9. Let X be a subtractive S-subsemimodule of a right S-semimodule Y, then the canonical projection $\pi_X : Y \to Y/X$ is k-quasiregular.

Proof. Since X is subtractive $Ker\ \pi_X = X$. Assume $K \leq Y$, $c \in Y \backslash K$, $c' \in K$, and $\pi_X(c) = \pi_X(c')$ then $c + X = c' + X$, but $c = c + 0 \in c + X = c' + X$, so, there exists $x \in X = Ker\ \pi_X$ such that $c = c' + x$. Thus, the canonical projection $\pi_X : Y \to Y/X$ is k-quasiregular. □

Prposition 2.10. If V is a subtractive S-subsemimodule of a right S-semimodule U and $V \ll U$, then U and U/V have the same projective cover.

Proof. $\Rightarrow$ Assume $\mathcal{P}$ is projective cover of U, hence, $f : \mathcal{P} \to U$ is coessential and $\mathcal{P}$ is projective, want to show that P is projective cover of U/V. Let $g = \pi_V \circ f : \mathcal{P} \to U/V$ Since $\mathcal{P}$ is projective enough to show that $g : \mathcal{P} \to U/V$ is coessential, so, for any S-homomorphism $h : C \to \mathcal{P}$ assume that $g \circ h$ is surjective want to show that h is surjective, $g \circ h = \pi_V \circ (f \circ h)$. Since $g \circ h$ surjective and π_V is superfluous by (Proposition 2.8), then $f \circ h$ is surjective by (Corollary 2.7), but f is coessential, therefore, $f \circ h$ is surjective only when h is surjective. Thus, g is coessential. So, $\mathcal{P}$ is projective cover of U/V.

$\Leftarrow$ Now, Assume that the converse is true, hence, $g : \mathcal{P} \to U/V$ is coessential and $\mathcal{P}$ is projective, want to show that $\mathcal{P}$ is projective cover of U, by projectivity of $\mathcal{P}$ there exists a homomorphism $f : \mathcal{P} \to U$ such that the diagram



commutes that is, $g = \pi_V \circ f$, since $\mathcal{P}$ is projective enough to show that $f : \mathcal{P} \to U$ is coessential, so, for any S-homomorphism $h : C \to \mathcal{P}$ assume that $f \circ h$ is surjective want to show h is surjective, but $g \circ h = \pi_V \circ (f \circ h)$, then $g \circ h$ is surjective, because π_V and $f \circ h$ are surjective, but g is coessential, therefore, $g \circ h$ is surjective only when h is surjective, so f is coessential. Consequently, $\mathcal{P}$ is projective cover of U. $\square$

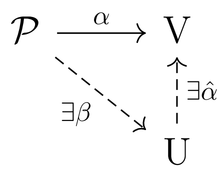
Prposition 2.11. If $p_t : P_t(U_t) \twoheadrightarrow M_t (t = 1, \ldots, n)$ are projective covers, then $\oplus \sum_{t=1}^{n} p_t : \oplus \sum_{t=1}^{n} P_t(U_t) \twoheadrightarrow \oplus \sum_{t=1}^{n} U_t$ is a projective cover.

Proof. If $p_t : P_t(U_i) \twoheadrightarrow U_t (t = 1, \ldots, n)$ are projective covers then each $P_t(U_t)$ is projective and so $\oplus \sum_{t=1}^{n} P_t(U_t)$ is projective by (Proposition 1.51). Also each $p_t : P_t(U_t) \twoheadrightarrow U_t (t = 1, \ldots, n)$ is coesential so for any $h_t : C_t \to P_t(U_t)(t = 1, \ldots, n)$, $p_t \circ h_t : C_t \to U_t$ is surjective only when h_t is surjective. Let $\oplus \sum_{t=1}^{n} p_t : \oplus \sum_{t=1}^{n} P_t(U_t) \to \oplus \sum_{t=1}^{n} U_t$ this is a surjective S-homomorphism since $\mathrm{Im}(\oplus \sum_{t=1}^{n} p_t) = \oplus \sum_{t=1}^{n} \mathrm{Im}(p_t) = \oplus \sum_{t=1}^{n} U_t$ since each p_t is surjective. Now we want to show that $\oplus \sum_{t=1}^{n} p_t : \oplus \sum_{t=1}^{n} P_t(U_t) \twoheadrightarrow \oplus \sum_{t=1}^{n} U_t$ is coessential so for any S-homomorphism $\oplus \sum_{t=1}^{n} h_t : \oplus \sum_{t=1}^{n} C_t \to \oplus \sum_{t=1}^{n} P_t(U_t)$ if $\oplus \sum_{t=1}^{n} p_t \circ \oplus \sum_{t=1}^{n} h_t : \oplus \sum_{t=1}^{n} C_t \to \oplus \sum_{t=1}^{n} U_t$ is surjective want to show that $\oplus \sum_{t=1}^{n} h_t$ is surjective, since $\oplus \sum_{t=1}^{n} p_t \circ \oplus \sum_{t=1}^{n} h_t$ is surjective then $\oplus \sum_{t=1}^{n} U_t = \mathrm{Im}(\oplus \sum_{t=1}^{n} p_t \circ \oplus \sum_{t=1}^{n} h_t) = \mathrm{Im}(\oplus \sum_{t=1}^{n} p_t \circ h_t) = \oplus \sum_{t=1}^{n} \mathrm{Im}\ (p_t \circ h_t)$ and so $U_t = \mathrm{Im}(p_t \circ h_t)$ and so, $p_t \circ h_t$ is surjective for each $(t = 1 \ldots n)$, then $h_t(t = 1 \ldots n)$ is surjective since each $p_t(t = 1 \ldots n)$ is coessential. Now $\mathrm{Im}(\oplus \sum_{t=1}^{n} h_t) = \oplus \sum_{t=1}^{n} \mathrm{Im}(h_t) = \oplus \sum_{t=1}^{n} P_t(U_t)$ since h_t is surjective for each $(t = 1, \ldots, n)$ and so $\oplus \sum_{t=1}^{n} h_t$ is surjective. Thus $\oplus \sum_{t=1}^{n} P_t : \oplus \sum_{t=1}^{n} P_t(U_t) \twoheadrightarrow \oplus \sum_{t=1}^{n} U_t$ is coessential. □

2.2 Weak Projectivity in Semiring Theory

We present in this section the concept of weak relative projectivity of right S-semimodule and we study properties related to the concept of weak projective semimodule in semiring theory corresponding to ring theory given in [13]. We check weak projectivity for some semimodules that have relations and study the closeness of this concept that is under what conditions (finite direct sum, subtractive subsemimodules ... etc) weak projectivity is closed.

Definition 2.12. For a semimodules U and V assume U has a projective cover $\sigma : \mathcal{P} \to U$. U **is weak V-projective** if for every map $\alpha : \mathcal{P} \to V$ there exists a surjective steady S-homomorphism $\beta : \mathcal{P} \to U$ and a homomorphism $\hat{\alpha} : U \to V$ in which the following diagram



commutes, that is $\alpha = \hat{\alpha}\beta$.

Definition 2.13. A semimodule U is weak projective if U is weak V-projective for each finitely generated right S-semimodule V.

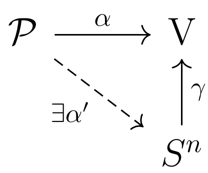
Theorem 2.14. For semimodules U and V assume U has a projective cover $\pi : \mathcal{P} \to U$. Then U is weak V-projective if for each $\alpha : \mathcal{P} \to V$ there exists a subtractive subsemimodule $K \subset \ker\alpha$ in which $\mathcal{P}/K \cong U$.

Proof. $\Rightarrow$ Consider the homomorphism $\alpha : \mathcal{P} \to V$. Since U is weak V projective the homomorphisms $\hat{\alpha} : U \to V$ and $\gamma : \mathcal{P} \to U$ exists and $\alpha = \hat{\alpha}\gamma$, and so $\ker\,\gamma \subset \ker\,\alpha$. Also, $\mathcal{P}/\ker\,\gamma \cong U$ by (Remark 1.36). Therefore, this direction is shown by choosing $K = \ker\,\gamma$ (kernels are subtractive subsemimodules).

$\Leftarrow$ Conversely, let $\alpha : \mathcal{P} \to V$ be a homomorphism and assume that there is $K \subset \mathcal{P}$ such that $\mathcal{P}/K \cong U$, then there is a surjective homomorphism $\gamma : \mathcal{P} \to U$ which yields from the composition of the isomorphism, $\beta : \mathcal{P}/K \to U$ and the natural projection $\pi_K : \mathcal{P} \to \mathcal{P}/K$, that is $\gamma = \beta \circ \pi_K$, and so $\ker\,\gamma = \ker\,\beta \circ \pi_K = \ker\,\pi_K = K \subset \ker\,\alpha$, but γ is steady homomorphism since if p_1 and p_2 are elements of $\mathcal{P}$ satisfying $\gamma(p_1) = \gamma(p_2)$ then $\beta \circ \pi_K(p_1) = \beta \circ \pi_K(p_2)$ and so $\pi_K(p_1) = \pi_K(p_2)$ then $p_1 + K = p_2 + K$. It follows by (Proposition 1.39) that there is $\hat{\alpha} : U \to V$ defined by $\hat{\alpha}(u) = \alpha(p)$, whenever $\gamma(p) = u$ which is well defined and $\alpha = \hat{\alpha}\gamma$. $\square$

Theorem 2.15. A right semimodule U is weak projective if and only if U is weak S^n-projective for each $n \in \mathcal{Z}^+$.

Proof. Just the converse direction needed to be shown. Let V be a finitely generated semimodule where $\alpha : \mathcal{P} \to V$. Since V is finitely generated, there is a surjective homomorphism $\gamma : S^n \to V$ for some $n \in \mathcal{Z}^+$. Since $\mathcal{P}$ is projective so there exists a homomorphism $\alpha' : \mathcal{P} \to S^n$ in which the diagram



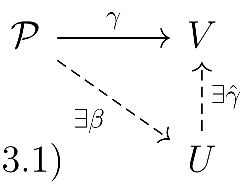
commutes, that is $\alpha = \gamma\alpha'$. Now, U is weak S^n-projective. So, there is $K \subset ker\alpha'$ such that $\mathcal{P}/K \cong U$ but $ker\alpha' \subset ker\alpha$. Therefore, $K \subset ker\alpha$ and so by previous theorem U is weak V-projective. □

Prposition 2.16. For semimodules U and V where V is subtractive and U assumed to have a projective cover $\sigma : \mathcal{P} \to U$. Next statements are equivalent:

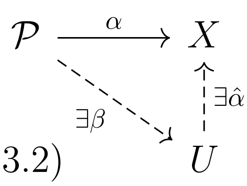
1. U is weak V-projective,
2. If X is subtractive subsemimodule of V then U is weak projective relative to X,
3. If X is subtractive subsemimodule of V then U is weak projective relative to V/X.

Proof. Since (2) and (3) gives (1) trivially, we want just to prove that (1) gives (2) and (3).

(1) $\Rightarrow$ (2) Let condition (1) assumed to be true and let X be a subtractive subsemimodule of V and $\alpha : \mathcal{P} \to X$ a homomorphism, there exists a map $\gamma = i_X \alpha : \mathcal{P} \to V$ since U is weak projective relative to V, there is a homomorphism $\hat{\gamma} : U \to V$ and steady surjective S-homomorphism $\beta : \mathcal{P} \to U$ such that $\gamma = \hat{\gamma}\beta$, that is diagram (3.1) commutes,for some homomorphism. Since β is a surjective steady homomorphism and $Ker(\beta) \subseteq Ker(\gamma)$, the range of $\hat{\gamma}(U) = \gamma(\mathcal{P})$ by (Proposition 1.39) and so $\hat{\gamma}(U)$ is contained in K. Now, since i_X is monic and $i_X(X) = X$ is a subtractive subsemimodule of X containing $\hat{\gamma}(U)$ by (Proposition 1.40) there exists a map $\hat{\alpha} : U \to X$ such that $\hat{\gamma} = i_X\hat{\alpha}$ but $i_X\alpha = \gamma = \hat{\gamma}\beta = (i_X\hat{\alpha})\beta$ then $\alpha = \hat{\alpha}\beta$ and then diagram (3.2) commutes. Therefore, U is weak X-projective.



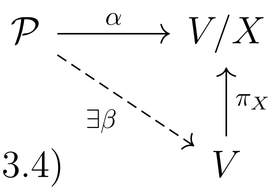
(3.1)



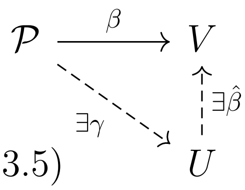
(3.2)

(1) $\Rightarrow$ (3) Another time assume that condition (1) is true and let $\alpha : \mathcal{P} \to V/X$ be a homomorphism. Now, projectivity of $\mathcal{P}$ yields that the map $\beta : \mathcal{P} \to V$ exists where diagram (3.4) commutes, that is $\alpha = \pi_X\beta$. Since U is weak projective relative to V there is a surjective steady homomorphim $\gamma : \mathcal{P} \to U$ and a homomorphism $\hat{\beta} : U \to V$ such that diagram (3.5) commutes, that is $\beta = \hat{\beta}\gamma$. Let $\hat{\alpha} = \pi_X\hat{\beta}$. Then $\hat{\alpha}\gamma = \pi_X\hat{\beta}\gamma = \pi_X\beta = \alpha$,

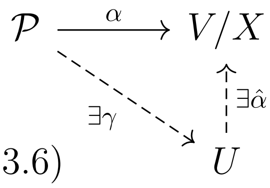
that is diagram (3.6) commutes and so U is weak V/X projective.



(3.4)



(3.5)



(3.6)

□

Prposition 2.17. For semimodules U and V where U is supplemented and has a projective cover $\pi : \mathcal{P} \to U$, if U is weak V-projective then for every subtractive subsemimodule $X \subset V$ and for every k-quasiregular surjective homomorphism $\alpha : \mathcal{P} \to X$, there is a surjective homomorphism $\hat{\alpha} : U \to X$ in which for every supplement Y' of $Ker\hat{\alpha}$ in U where Y' subtractive proper subsemimodule of U there is a subsemimodule $Y \subset \mathcal{P}$, where $\mathcal{P}/Y \simeq U/Y'$ and $Y + Ker\alpha = \mathcal{P}$.

Proof. Let U assumed to be weak projective relative to V and $\alpha : \mathcal{P} \to V$ be a surjective homomorphism. So, there is a surjective homomorphism $\gamma : \mathcal{P} \to U$ and a homomorphism $\hat{\alpha} : U \to X$ in which $\alpha = \hat{\alpha}\gamma$. Let Y' be a supplement of $Ker\hat{\alpha}$ in U and $Y = \gamma^{-1}(Y')$. Now, for any $p \in \mathcal{P}$, we can write $\gamma(p)$ as $\gamma(p) = y' + e$, where $y' \in Y'$ and $e \in Ker\hat{\alpha}$. But, $\alpha(p) = \hat{\alpha}\gamma(p) = \hat{\alpha}(y') + \hat{\alpha}(e) = \hat{\alpha}(y')$. Now, choose $p_1 \in \gamma^{-1}(y') \subset Y$, then $\gamma(p_1) = y'$. Also, $\alpha(p_1) = \hat{\alpha}\gamma(p_1) = \hat{\alpha}(y') = \alpha(p)$. Since α is k-quasiregular so there is $e_1 \in Ker\alpha$ such that $p_1 + e_1 = p$ and so $Y + Ker\alpha = \mathcal{P}$. Also, $\mathcal{P}/Y \simeq U/Y'$ since Y is the kernel of the surjective map $\pi_{Y'}\gamma : \mathcal{P} \to U/Y'$. □

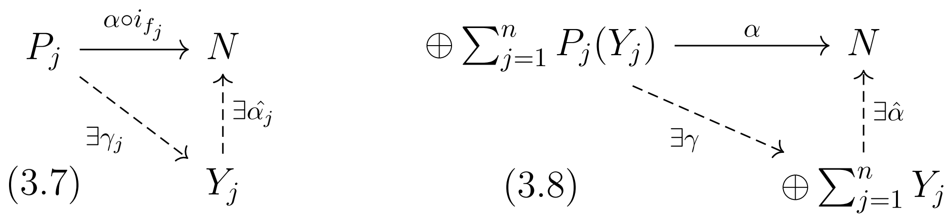
Prposition 2.18.

1. Let $Y_j, j = 1, 2, \ldots n$ be a set of weak V-projective semimodules. Then $\oplus_{j=1}^{n} Y_j$ is weak V-projective.

2. If Y/X is weak K-projective semimodule and X is a subtractive S-subsemimodule of a right S-semimodule Y with $X \ll Y$. Then Y is weak projective relative to K.

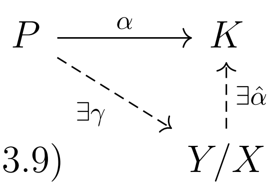
3. If a semimodule Y is weak $\mathcal{P}(Y)$ projective, then the semimodule Y is projective.

Proof. (1) Let $f_j : P_j(Y_j) \to Y_j$, (j = 1,...,n) be projective covers. By (Proposition 2.11), $\oplus \sum_{i=1}^{n} f_j : \oplus \sum_{j=1}^{n} P_j(Y_j) \to \oplus \sum_{i=1}^{n} Y_j$ is a projective cover. Let $\alpha : \oplus \sum_{j=1}^{n} P_j(Y_j) \to N$, and let $i_{f_j} : P_j(Y_j) \to \oplus \sum_{j=1}^{n} P_j(Y_j)$ be the inclusion map. Since $Y_j's$ are weakly projective for each j, there exists a surjective steady homomorphism $\gamma_j : P_j(Y_j) \to Y_j$ and $\hat{\phi}_j : Y_j \to N$ such that diagram (3.7) commutes, that is $\hat{\phi}_j \gamma_j = \alpha_j i_{f_j}$. Set $\hat{\alpha} = \oplus \sum_{j=1}^{n} \hat{\alpha}_j$ and $\gamma = \oplus \sum_{j=1}^{n} \gamma_j$. Then diagram (3.8) commutes, that is $\alpha = \hat{\alpha}\gamma$.

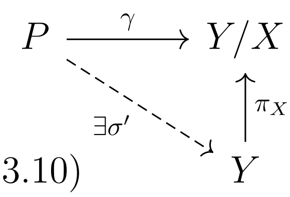


(2) Since X is a subtractive S-subsemimodule of a right S-semimodule Y and $X \ll Y$. By (Proposition 2.10), $P(Y) = P(Y/X)$ (their projective cover

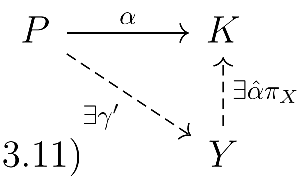
is the same). Let $\alpha : P(Y) \to K$ and $\pi_X : Y \to Y/X$ be the natural projection. A steady homomorphism $\gamma : P(Y) \to Y/X$ and a homomorphism $\hat{\alpha} : Y/X \to K$ are exists because Y/X is weakly K-projective, such that diagram (3.9) commutes, that is $\hat{\alpha}\gamma = \alpha$. Now, because $P(Y)$ is projective, there is $\gamma' : P(Y) \to Y$ where diagram (3.10) commutes, that is $\pi_X\gamma' = \gamma$. Because $X \ll Y$ and X is subtractive, it follows by (Corollary 2.7) that γ' is surjective also γ' is steady homomorphism since if p_1 and p_2 are elements of P satisfying $\gamma'(p_1) = \gamma'(p_2)$ then $\pi_X\gamma'(p_1) = \pi_X\gamma'(p_2)$ and so $\gamma(p_1) = \gamma(p_2)$ by steadiness of γ there exists e_1, $e_2 \in Ker\gamma$ such that $p_1 + e_1 = p_2 + e_2$, but e_1, $e_2 \in Ker\gamma'$ that is $\gamma'(e_1) = \gamma'(e_1) = 0$ if not, that is $\gamma'(e_1) = a$ where $a \neq 0$ then $\pi_X\gamma'(e_1) = \pi_X(a)$ and so $\gamma(e_1) = a + X \neq 0 + X$ this implies that $e_1 \notin Ker\gamma$ and this contradicts our assumption so $e_1 \in Ker\gamma'$ similarly for e_2 . We can easily check that diagram (3.11) commutes, that is $\hat{\phi}\pi_X\gamma' = \phi$. Therefore, Y is weakly K- projective.



(3.9)

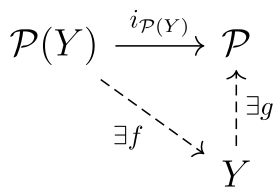


(3.10)



(3.11)

(3) Consider a semimodule Y and it's projective cover is $\gamma : \mathcal{P}(Y) \to Y$. If Y is weak $\mathcal{P}(Y)$-projective, then the identity map on $\mathcal{P}(Y)$ factors through Y



$i_{\mathcal{P}(Y)} = gf$ and $i_Y = fg$, this yields that $Y \cong \mathcal{P}(Y)$ and $\mathcal{P}(Y)$ is projective, therefore Y is projective. $\square$

CHAPTER 3

WEAKLY INJECTIVE SEMIMODULES

We dualize in this chapter some of the results of weak projectivity studied in the previous chapter. In section 1, we state some basics about injective envelopes. In section 2, we introduce the definition of weak injectivie right S-semimodule and we study some properties related to this concept in semiring theory corresponding to ring theory.

3.1 Injective Hulls (Envelopes) of Semimodules

We present in this section the concept of injective envelopes and state some of its basic properties and study the relation between injective envelops of some semimodules.

Definition 3.1. [11] An injective homomorphism $\gamma : X \to Y$ of S-semimodules is essential if and only if for any S-homomorphism $\xi : Y \to Z$, if the map $\xi \circ \gamma$ is injective then ξ is injective.

Definition 3.2. [11] A subsemimodule X is large in Y if and only if the inclusion map $i_X : X \to Y$ is an essential homomorphism. Moreover, if X is large in Y then, Y is called an essential extension of X.

Definition 3.3. [11] A semimodule H is an injective hull of a semimodule U if:

1. If H is an essential extension of U and,
2. H is injective.

If a semimodule U has an injective hull H, we denote it as $H(U)$. However, an arbitrary semimodule may have no injective hull.

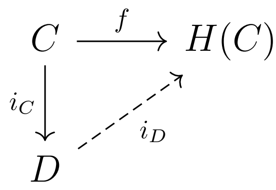
Remark 3.4. U is injective semimodule if and only if $U = H(U)$.

Proof. Immediate from the definition of the injective envelope. □

Prposition 3.5. Let C and D be semimodules and assume D has an injective hull. If D is an essential extension of C, then $H(D)$ is an essential extension of C.

Proof. Since D is an essential extension of C the inclusion $i_C : C \to D$ is essential, also since $H(D)$ is an essential extension of D the inclusion $i_D : D \to H(D)$ is essential want to show that the inclusion $f : C \to H(D)$ is essential. Assume not that is there is a homomorphism $g : H(D) \to K$ such that $g \circ f$ is injective but g is not since $i_D : D \to H(D)$ is essential and $g : H(D) \to K$ is not injective and so $g \circ i_D$ is not injective also since $i_C : C \to D$ is essential and $g \circ i_D : D \to K$ is not injective and so $(g \circ i_D) \circ i_C$

is not injective by injectivity of $H(D)$ the diagram



commutes that is, $f = i_D \circ i_C$ and so $g \circ f = g \circ (i_D \circ i_C)$ which is not injective which contradicts our hypothesis. Thus, f is essential and so $H(D)$ is an essential extension of C.

□

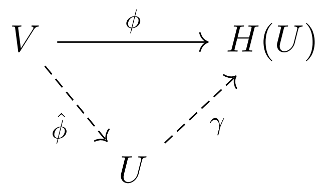
Prposition 3.6. Let A and B be semimodules and assume A has an injective hull. If B is an essential extension of A, then $H(A) = H(B)$.

Proof. Since B is an essential extension of A the previous proposition implies that $H(B)$ is an essential extension of A that is $H(B)$ is essential extension of A and $H(B)$ is injective, so $H(B)$ is an injective hull of A that is, $H(A) = H(B)$. □

3.2 Weak Injectivity in Semiring Theory

In this section we introduce the concept of weak relative injectivity of right S-semimodule and study some properties related to the concept in semiring theory corresponding to ring theory given in [12]. We check weak injectivity for some semimodules that have relations and study the closeness of this concept that is under what conditions (finite direct sum, subtractive subsemimodules . . . etc) weak injectivity is closed.

Definition 3.7. For a semimodules U and V assume U has an injective hull ξ: $U \to H$. U **is weak V-injective** if and only if for every homomorphism $\phi : U \to H(V)$ there is a monomorphism (injective S-homomorphism) $\gamma : U \to H(U)$ and a homomorphism $\hat{\phi} : V \to U$ in which the diagram



commutes, that is $\phi = \gamma\hat{\phi}$.

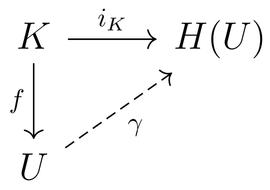
Definition 3.8. A semimodule U is weak injective if it is weak V-injective for every finitely generated semimodule V.

Theorem 3.9. For a semimodule U which has an injective hull $\xi : U \to H(U)$. U is weak V-injective if for every homomorphism $\alpha : V \to H(U)$, $\alpha(V) \subset K \cong U$, for some subsemimodule K of $H(U)$.

Proof. $\Rightarrow$ Let $\alpha : V \to H(U)$. Since U is weak V-injective there exists homomorphisms $\gamma : U \to H(U)$ and $\hat{\alpha} : V \to U$ in which $\alpha = \gamma\hat{\alpha}$ where γ is monomorphism, and so $\alpha(V) = \gamma\widehat{\alpha(V)} \subset \gamma(U) \cong U$, so this direction is shown by choosing $K = \gamma(U)$ where $\gamma(U)$ is subsemimodule of $H(U)$.

$\Leftarrow$ Conversely, Assume that there is K satisfying that $\alpha(V) \subset K \cong U$, then there is an isomorphism $f : K \to U$ also consider the inclusion $i_K : K \to$

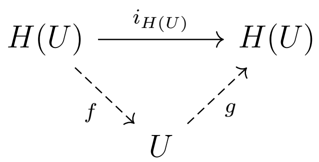
$H(U)$ then sine $H(U)$ is injective



there is a monomorphism $\gamma : U \to H(U)$. Now we have $\alpha : V \to H(U)$ and $\gamma : U \to H(U)$ in which $\alpha(V) \subset U \cong \gamma(U)$ and $\gamma(U)$ subtractive since $\gamma(U) \cong U$ and U subtractive since f is surjective by (Proposition 1.25(3)). Now using (Proposition 1.40) there is a homomorphism $\hat{\alpha} : V \to U$ satisfying $\alpha = \gamma\hat{\alpha}$ proving that U is weakly V-projective. □

Prposition 3.10. Let U be a semimodule and assume U has an injective hull. U is injective if U is weak $H(U)$ injective.

Proof. Consider a semimodule U with injective hull $\gamma : U \to H(U)$. If we assume that U is weak $H(U)$-injective, then the identity map on $H(U)$ factors through U



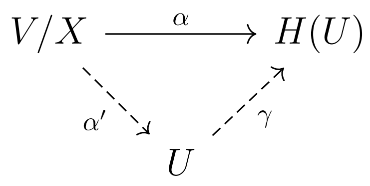
$i_{H(U)} = gf$ and $i_U = fg$ this yields that $U \cong H(U)$ and $H(U)$ is injective. Thus, U is injective. □

Prposition 3.11. For semimodules U and V assume U has an injective hull $\xi : U \to H(U)$. Next statements are equivalent:

1. U is weak V-injective,

2. If X is subsemimodule of V then U is weakly V/X-injective,

3. If X is subsemimodule of V then for each monomorphism $\alpha : V/X \to H(U)$ there is monomorphisms $\gamma : U \to H(U)$ and $\alpha' : V/X \to U$ such that the diagram



commutes that is, $\alpha = \gamma\alpha'$.

Proof. (1) $\to$ (3). Since U is weak V-injective, let a homomorphism $\beta : V \to H(U)$ there are homomorphisms $\gamma : U \to H(U)$ and $\beta' : V \to U$ in which $\beta = \gamma\beta'$ where γ is a monomorphism . Now, consider the homomorphism $\alpha : V/X \to H(U)$ where $\alpha(v + X) = \beta(v)$, and define $\alpha' : V/X \to U$ as $\alpha'(v + X) = \psi'(v)$. Thus, $\alpha = \gamma\alpha'$.

(3) $\to$ (2). Trivially by definition.

(2) $\to$ (1). Since U is weak V/X injective, for each homomorphism $\alpha : V/X \to H(U)$ there is a monomorphism $\gamma : U \to H(U)$ and a homomorphism $\alpha' : V/X \to U$ in which $\alpha = \gamma\alpha'$. Let us define $\beta : V \to H(U)$ as $\beta(v) = \alpha(v + X)$ and $\beta'(v) = \alpha'(v + X)$, then $\beta = \gamma\beta'$. Thus, U is weak V-injective. □

Theorem 3.12. A semimodule U is weak injective if and only if U is weak S^n-injective for all $n \in \mathcal{Z}^+$.

Proof. Just the converse direction needed to be shown. Let V be a finitely generated semimodule where $\alpha : V \to H(U)$. Since V is finitely generated, there is a surjective homomorphism $\rho : S^n \to V$ for some $n \in \mathcal{Z}^+$. Since U is weak S^n-injective, so by (Theorem 3.9) for each homomorphism $\alpha\rho$:

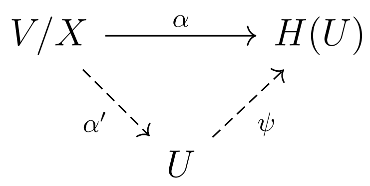
$S^n \to H(U)$, $\alpha\rho(S^n) \subset K \cong U$ for some subsemimodule K of $H(U)$, and so $\alpha(\rho(S^n)) \subset K \cong U$, but since ρ is surjective $\rho(S^n) = V$. Therefore, $\alpha(V) \subset K \cong U$ for some $K \leq H(U)$. Thus, U is weakly V injective. □

Prposition 3.13. For semimodules U and V, assume U has an injective hull $\xi : U \to H(U)$. U is weak V-injective if and only if for each subsemimodule X of V and for every monomorphism $\alpha : V/X \to H(U)$:

1. There exists a monomorphism $\alpha' : V/X \to U$,
2. If C is complement of $\alpha'(V/X)$ in U, where C and $\alpha'(V/X)$ are subtractive subsemimodules of U, there exist $C' \subseteq H(U)$ such that $C' \cap \alpha(V/X) = 0$ and $X' \cong X$.

Proof. $\Rightarrow$ Let $\alpha : V/X \to H(U)$ be a monomorphism. Using Proposition 3.11(3), there are homomorphisms $\gamma : U \to H(U)$ and $\alpha' : V/X \to U$ in which $\alpha = \alpha\gamma'$ where γ is monomorphism. first part is done. Now for (2); Let C be a complement of $\alpha'(V/X)$ in U, then $C' = \gamma(C) \cong C$ and $C' \cap \alpha(V/X) = 0$, if not there exists $0 \neq x \in C' \cap \alpha(V/X)$ and so, $0 \neq \gamma^{-1}(x) \in \gamma^{-1}(C') \cap \gamma^{-1}(\alpha(V/X)) = C \cap \alpha'(V/X)$ a contradiction. Thus $C' \cap \alpha(V/X) = 0$.

$\Leftarrow$ Conversely, let $\alpha : V/X \to H(U)$ be a monomorphism, there is $\alpha' : V/X \to U$ by (1). Now, Let T be a complement of $\alpha'(V/X)$ in U, then $\alpha'(V/X) \oplus T$ is large in U and so $H(\alpha'(V/X) \oplus T) = H(U)$ so this gives a monomorphism $\gamma : \alpha'(V/X) \oplus T \to H(U)$. Since $\alpha'(V/X) \oplus T \trianglelefteq U$ this yields a homomorphism $\alpha'(V/X) \oplus T \to U$ by injectivity of $H(U)$, γ can be extended to a monomorphism $\psi : U \to H(U)$. It is immediate that $\psi\alpha' = \alpha$, that is the diagram



commutes. Then, by Proposition 3.11(3) U is weakly V-injective. □

Prposition 3.14. 1. If X and Y are weak V-injective semimodules, then $X \oplus Y$ is weak V-injective.

2. If Y is weak V-injective and Y is large in X, then X is weak V-injective.

Proof. 1. Since X is weakly V-injective. Then by (Theorem 3.9) for each homomorphism $\psi_1 : V \to H(X)$, $\psi_1(V) \subset K \cong X$ for some subsemimodule K of $H(X)$. Also, Y is weak V-injective, so for each homomorphism $\psi_2(V) \subset L \cong Y$ for some submodule L of $H(Y)$. Let $\psi : N \to H(X) \oplus H(Y)$. Then $\psi(V) = \psi_1(V) \oplus \psi_2(V) \subset K \oplus L \cong X \oplus Y$ hence $X \oplus Y$ is weak V-injective.

2. Since Y large in X, then $H(Y) = H(X)$ by (Proposition 3.6) X is weak V-injective.

□

CONCLUSION

In this thesis we generalized the concept of weakly projective module for semimodule over semiring which is weakly projective semimodule and we studied some of its basic characteristics which are analogous to ring theory, also we dualize the concept of weakly projective semimodule for weakly injective semimodule and we studied its basic properties in a similar manner.

FUTURE WORK

It could be interesting to study the relations between the concepts of weakly projective and weakly injective semimodules and under what conditions a relation may exists.

When weakly projective semimodule is weakly injective and the converse?

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